TWISTED FIRST HOMOLOGY GROUP OF THE AUTOMORPHISM GROUP OF A FREE GROUP (Perspectives of Hyperbolic Spaces II)

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TWISTED FIRST HOMOLOGY GROUP OF THE AUTOMORPHISM GROUP OF A FREE GROUP

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Abstract: The automorphism group $\mathrm{Aut}\, F_n$ and the outer automorphism group $\mathrm{Out}\, F_n$ of a free group F_n of rank n act on the abelianized group H of F_n and the dual group H^* of H. The twisted first homology groups of $\mathrm{Aut}\, F_n$ and $\mathrm{Out}\, F_n$ with coefficients in H and H^* are calculated.

Keywords: automorphism group of a free group, mapping class group, Magnus representation

1. INTRODUCTION

Let F_n be a free group of rank n and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. There are remarkable results of the homology groups of $\mathrm{Aut}\, F_n$ with trivial coefficients. For example, Gersten [2] showed $H_2(\mathrm{Aut}\, F_n, \mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ for $n \geq 5$ and Hatcher and Vogtmann [3] showed $H_i(\mathrm{Aut}\, F_n, \mathbf{Q}) = 0$ for $n \geq 1$ and $1 \leq i \leq 6$, except for $H_4(\mathrm{Aut}\, F_4, \mathbf{Q}) = \mathbf{Q}$. However, there are very few computations of twisted homology groups of $\mathrm{Aut}\, F_n$.

Fix a free basis Y of F_n. Since the abelianized group H of F_n is isomorphic to $\mathbf{Z}^n$, abelianization induces a homomorphism $\varphi : \mathrm{Aut}\, F_n \twoheadrightarrow \mathrm{Aut}\, H = GL(n, \mathbf{Z})$. The map φ induces the action of $\mathrm{Aut}\, F_n$ on H, and hence the dual group $H^* = \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ of H. We denote by $\mathrm{Out}\, F_n$ the outer automorphism group of F_n. Since φ induces a natural map $\bar{\varphi} : \mathrm{Out}\, F_n \twoheadrightarrow GL(n, \mathbf{Z})$, $\mathrm{Out}\, F_n$ also acts on H and H^*. In this paper, we calculate the twisted first homology groups of $\mathrm{Aut}\, F_n$ and $\mathrm{Out}\, F_n$ with coefficients in H and H^*. Let $\det : GL(n, \mathbf{Z}) \longrightarrow \{\pm 1\}$ be the determinant map. The groups $\mathrm{Aut}^+ F_n = \ker(\det \circ \varphi)$ and $\mathrm{Out}^+ F_n = \ker(\det \circ \bar{\varphi})$ are called the special automorphism group and the special outer automorphism group of F_n respectively. The following theorem is our main result.

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Theorem 1. *For $n \geq 2$, we have:*

(1) *If* $\Gamma_n = \operatorname{Aut} F_n$ *or* $\operatorname{Aut}^+ F_n$,

$$H_1(\Gamma_n, H) = \begin{cases} 0 & \text{if } n \geq 4, \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 3, \\ \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } n = 2 \text{ and } \Gamma_2 = \operatorname{Aut} F_2, \\ \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } n = 2 \text{ and } \Gamma_2 = \operatorname{Aut}^+ F_2, \end{cases}$$

$$H_1(\Gamma_n, H^*) = \begin{cases} \mathbf{Z} & \text{if } n \geq 4, \\ \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } n = 2, 3. \end{cases}$$

(2) *If* $\Omega_n = \operatorname{Out} F_n$ *or* $\operatorname{Out}^+ F_n$,

$$H_1(\Omega_n, H) = \begin{cases} 0 & \text{if } n \geq 4, \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 2, 3, \end{cases}$$

$$H_1(\Omega_n, H^*) = \begin{cases} \mathbf{Z}/(n-1)\mathbf{Z} & \text{if } n \geq 4, \\ \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } n = 3, \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 2. \end{cases}$$

In Section 2, we introduce Gersten's finite presentation for $\operatorname{Aut}^+ F_n$. We simplify his presentation using Titze transformations. We use it to calculate the first cohomology group of $\operatorname{Aut}^+ F_n$.

In Section 5, we give some consequences of our results. We show that the generator of $H^1(\operatorname{Aut}^+ F_n, H) = \mathbf{Z}$ is induced by the Magnus representation of $\operatorname{Aut}^+ F_n$. This shows that the natural map $M_{g,1} \hookrightarrow \operatorname{Aut}^+ F_{2g}$ induces an isomorphism $H^1(\operatorname{Aut}^+ F_{2g}, H) \to H^1(M_{g,1}, H)$ where $M_{g,1}$ is the mapping class group of a surface of genus g with one boundary component.

2. A PRESENTATION FOR THE SPECIAL AUTOMORPHISM GROUP OF A FREE GROUP

In this section, we introduce Gersten's finite presentation for $\operatorname{Aut}^+ F_n$. Let $Y = \{y_1, \dots, y_n\}$ be a free basis of F_n and let $Y^{\pm 1} = \{y \mid y \text{ or } y^{-1} \in Y\}$. For any a, $b \in Y^{\pm 1}$ with $a \neq b^{\pm 1}$, difine the Nielsen automorphism E_{ab} by the rule $a \mapsto ab$, $c \mapsto c$ if $c \in Y^{\pm 1} \setminus \{a^{\pm 1}\}$ and let $w_{ab} = E_{ba} E_{a^{-1}b} E_{b^{-1}a^{-1}}$. The map w_{ab} induces a permutation σ of $Y^{\pm 1}$ $a \mapsto b^{-1}$, $b \mapsto a$, called the monomial map determined by w_{ab}. Gersten [2] showed that $\operatorname{Aut}^+ F_n$ has a following presentation.

Theorem 2.1 (Gersten [2]). *For $n \geq 3$, a presentation for* $\operatorname{Aut}^+ F_n$ *is given by the generators E_{ab} and relations:*

(R1): $E_{ab}{}^{-1} = E_{ab^{-1}}$,

(R2): $[E_{ab}, E_{cd}] = 1,\ a \neq c, d^{\pm 1},\ b \neq c^{\pm 1}$,
(R3): $[E_{ab}, E_{bc}] = E_{ac},\ a \neq c^{\pm 1}$,
(R4): $w_{ab} = w_{a^{-1}b^{-1}}$
(R5): $w_{ab}{}^4 = 1$.

Here $[\, , \,]$ *denotes the commutator bracket:* $[x, y] = xyx^{-1}y^{-1}$.

Remark 2.1. *Gersten* [2] *also showed that if* $n = 2$*, the group* Aut^+F_2 *has a presentation which is given by the generators* E_{ab} *subject to the relations* (R1) – (R3), (R5) *and*

$$(\mathrm{R4})' : w_{ab}{}^{-1}E_{cd}w_{ab} = E_{\sigma(c)\sigma(d)},$$

where σ *is the monomial map determined by* w_{ab}.

Using Titze transformations, we have the following presentaton for Aut^+F_n for $n \geq 3$.

Theorem 2.2. *For* $n \geq 3$*, a presentation for* Aut^+F_n *is given by the generators* $E_{y_iy_j}$ *and* $E_{y_i^{-1}y_j}$ *subject to the relations:*

(R2-1): $[E_{y_iy_j}, E_{y_i^{-1}y_j}] = 1$,
(R2-2): $[E_{y_iy_j}, E_{y_ky_j}] = 1$,
(R2-3): $[E_{y_i^{-1}y_j}, E_{y_ky_j}] = 1$,
(R2-4): $[E_{y_i^{-1}y_j}, E_{y_k^{-1}y_j}] = 1$,
(R2-5): $[E_{y_iy_j}, E_{y_i^{-1}y_k}] = 1$,
(R2-6): $[E_{y_iy_j}, E_{y_ky_l}] = 1$,
(R2-7): $[E_{y_i^{-1}y_j}, E_{y_ky_l}] = 1$,
(R2-8): $[E_{y_i^{-1}y_j}, E_{y_k^{-1}y_l}] = 1$,
(R3-1): $[E_{y_iy_k}, E_{y_ky_j}] = E_{y_iy_j}$,
(R3-2): $[E_{y_iy_k^{-1}}, E_{y_k^{-1}y_j}] = E_{y_iy_j}$,
(R3-3): $[E_{y_i^{-1}y_k}, E_{y_ky_j}] = E_{y_i^{-1}y_j}$,
(R3-4): $[E_{y_i^{-1}y_k^{-1}}, E_{y_k^{-1}y_j}] = E_{y_i^{-1}y_j}$,
(R4-1): $w_{y_iy_j} = w_{y_i^{-1}y_j^{-1}}$,
(R5-1): $w_{y_iy_j}{}^4 = 1$,

where $E_{y_iy_j^{-1}}$ *is understood to be* $E_{y_iy_j}{}^{-1}$.

3. THE AUTOMORPHISM GROUP OF A FREE GROUP

Until Section 4, we assume $n \geq 3$. For any integer $q \geq 2$, let $A_q = H \otimes_{\mathbf{Z}} (\mathbf{Z}/q\mathbf{Z})$ and $A_q^* = H^* \otimes_{\mathbf{Z}} (\mathbf{Z}/q\mathbf{Z})$. Let $M = H$, H^*, A_q or A_q^*. Using the presentation for Aut^+F_n obtained by Theorem 2.2, we can calculate the twisted first cohomology groups of Aut^+F_n as follows:

Proposition 3.1. *Let $q \geq 2$ and $e \geq 1$ be positive integers. For $n \geq 3$, we have*

$$H^1(\mathrm{Aut}^+F_n, H) = \mathbf{Z},$$
$$H^1(\mathrm{Aut}^+F_n, A_q) = \begin{cases} \mathbf{Z}/q\mathbf{Z} & \text{if } (q,2) = 1, \\ \mathbf{Z}/q\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } n = 3 \text{ and } q = 2^e. \end{cases}$$

Proposition 3.2. *Let $q \geq 2$ and $e \geq 1$ be positive integers. For $n \geq 3$, we have*

$$H^1(\mathrm{Aut}^+F_n, H^*) = 0,$$
$$H^1(\mathrm{Aut}^+F_n, A_q^*) = \begin{cases} 0 & \text{if } (q,2) = 1, \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 3 \text{ and } q = 2^e. \end{cases}$$

Observing the spectral sequence of the group extension

$$1 \to \mathrm{Aut}^+F_n \to \mathrm{Aut}\,F_n \to \{\pm 1\} \to 1,$$

we see that $H^1(\mathrm{Aut}\,F_n, M) \simeq H^1(\mathrm{Aut}^+F_n, M)$ For $M = H$, H^*, A_q or A_q^*. Then, using the universal coefficient theorem, we obtain the twisted first homology groups of $\mathrm{Aut}\,F_n$.

4. The outer automorphism group of a free group

Let $\mathrm{Inn}\,F_n$ be the group of inner automorphisms of F_n. Observing the spctral sequence of the group extension

$$1 \to \mathrm{Inn}\,F_n \to \mathrm{Aut}^+F_n \to \mathrm{Out}^+F_n \to 1,$$

we calculate the twisted first cohomology groups of Out^+F_n as follows:

Proposition 4.1. *Let $q \geq 2$ and $e \geq 1$ be positive integers. For $n \geq 3$, we have*

$$H^1(\mathrm{Out}^+F_n, H) = 0, \quad H^1(\mathrm{Out}^+F_n, H^*) = 0.$$

Proposition 4.2. *Let $q \geq 2$ and $e \geq 1$ be positive integers. For $n \geq 3$, we have*

(1) *If $n = 3$,*

$$H^1(\mathrm{Out}^+F_3, A_q) = \begin{cases} 0 & \text{if } (q,2) = 1, \\ \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & \text{if } q = 2^e, \end{cases}$$
$$H^1(\mathrm{Out}^+F_3, A_q^*) = \begin{cases} 0 & \text{if } (q,2) = 1, \\ \mathbf{Z}/2\mathbf{Z} & \text{if } q = 2^e. \end{cases}$$

(2) *If* $n \geq 4$,

$$H^1(\mathrm{Out}^+F_n, A_q) = \begin{cases} 0 & \text{if } (q, n-1) = 1, \\ \mathbf{Z}/q\mathbf{Z} & \text{if } q \mid (n-1), \\ \mathbf{Z}/(n-1)\mathbf{Z} & \text{if } (n-1) \mid q, \end{cases}$$
$$H^1(\mathrm{Out}^+F_n, A_q^*) = 0.$$

Then, using the universal coefficient theorem, we obtain the twisted first homology groups of Out^+F_n. Furthermore, observing the spectral sequence of the group extension

$$1 \to \mathrm{Out}^+F_n \to \mathrm{Out}\,F_n \to \{\pm 1\} \to 1,$$

we see that $H^1(\mathrm{Out}\,F_n, M) \simeq H^1(\mathrm{Out}^+F_n, M)$ For $M = H$, H^*, A_q or A_q^*. Then, using the universal coefficient theorem, we obtain the twisted first homology groups of $\mathrm{Out}\,F_n$.

5. Some consequences

we show that the generator of $H^1(\mathrm{Aut}^+F_n, H) = \mathbf{Z}$ is induced by the Magnus representation of Aut^+F_n. For any generator y_j $(1 \leq j \leq n)$ of F_n, let

$$\frac{\partial}{\partial y_j} : \mathbf{Z}[F_n] \longrightarrow \mathbf{Z}[F_n]$$

be the Fox free derivatives. (See [1].) Let $^- : \mathbf{Z}[F_n] \to \mathbf{Z}[F_n]$ be the antiautomorphism induced from the map $F_n \ni y \mapsto y^{-1} \in F_n$. Then the Magnus representation $r : \mathrm{Aut}^+ F_n \longrightarrow GL(n, \mathbf{Z}[F_n])$ of Aut^+F_n is defined to be

$$r(\sigma) = \left(\overline{\frac{\partial \sigma(y_j)}{\partial y_i}}\right)_{(i,j)}.$$

Let $\sigma_* : \mathbf{Z}[F_n] \to \mathbf{Z}[F_n]$ be the automorphism of $\mathbf{Z}[F_n]$ induced from σ. The map r satisfies

$$r(\sigma\tau) = r(\sigma) \cdot r(\tau)^\sigma. \tag{1}$$

Here $r(\tau)^\sigma$ denotes the matrix obtained from $r(\tau)$ by applying σ_* on each entry. (See [5].) Let $a' : GL(n, \mathbf{Z}[F_n]) \to GL(n, \mathbf{Z}[H])$ be the homomorphism induced from the abelianizer $a : F_n \to H$ and det : $\mathrm{GL}(n, \mathbf{Z}[H]) \to \mathbf{Z}[H]$ the determinant homomorphism. Then we put

$$f_M = \det \circ a' \circ r : \mathrm{Aut}^+F_n \longrightarrow \mathbf{Z}[H].$$

Observing our results obtained in Section 3, we have

Lemma 5.1. *The map f_M is a crossed homomorphism from Aut^+F_n to H and generates $H^1(\mathrm{Aut}^+F_n, H)$.*

Remark 5.1. *We should remark that the same argument does not hold in the case $H^1(\mathrm{Aut}\,F_n, H)$. In this case, the image of the crossed homomorphism $f_M : \mathrm{Aut}\,F_n \to \mathbf{Z}[H]$ is not included in H.*

Morita [4] calculated $H^1(M_{g,1}, H_1(\Sigma_{g,1}, \mathbf{Z})) = \mathbf{Z}$ and show that the generator of $H^1(M_{g,1}, H_1(\Sigma_{g,1}, \mathbf{Z}))$ is also given by the Magnus representation of $M_{g,1}$. (See [5].) Hence we have

Corollary 5.1. *The natural map $M_{g,1} \hookrightarrow \mathrm{Aut}^+ F_{2g}$ induces an isomorphism*

$$\mathrm{res} : H^1(\mathrm{Aut}^+ F_{2g}, H) \to H^1(M_{g,1}, H_1(\Sigma_{g,1}, \mathbf{Z})).$$

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